Satisfying stress constraints in density based topology optimization by length scale control

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Difficulties with stress constraints

Characteristics of stress-constrained continuum topology optimization:

- Basic engineering requirement: remain linear-elastic, reduce stress concentrations
- Local measure $\rightarrow$ large number of constraints
- Removal of material $\rightarrow$ vanishing of constraint
Difficulties with stress constraints

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Challenge #1: COMPLEXITY

Large number of design variables, large number of constraints

Challenge #2: SINGULARITY

Difficult to capture true optimum by numerical procedures
Successful approaches for constraining stresses

- Consider all local constraints, solve with “active” subsets
- Aggregate local constraints into global stress function, using K-S or $p$-norm functions
- Apply external penalization on stress violations
- Employ nonlinear modeling or artificial damage

**Common to all approaches**: the stress / behavior constraint is a function of topological variables
Goal: study the role of length scale

We seek to **study the role of length scale**:

- Stress concentrations / violations are often related to length scale (thickness, curvature):

- Shape and sizing following topology may be able to deal with most issues, but creating a parametrized model can be painful:
Controlling the length scale

We follow density-based procedures so control of length scale is via filter radius and Heaviside projections:

- Well-known density filter (Bruns & Tortorelli 2001, Bourdin 2001)
How does length scale influence stresses?

The effect of filter radius, $\eta_d = 0.4$, $\eta_e = 0.6$:

$r_{\text{min}}$, LS $\uparrow$ compliance $\uparrow$ stress $\downarrow$

<table>
<thead>
<tr>
<th>$r_{\text{min}}$</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>LS</td>
<td>1.9</td>
<td>3.2</td>
<td>4.4</td>
<td>5.7</td>
</tr>
<tr>
<td>$f^T u$</td>
<td>$2.235 \cdot 10^2$</td>
<td>$2.322 \cdot 10^2$</td>
<td>$2.363 \cdot 10^2$</td>
<td>$2.445 \cdot 10^2$</td>
</tr>
<tr>
<td>$\sigma_{\text{VM}}^{\text{max}}$</td>
<td>$6.040 \cdot 10^{-1}$</td>
<td>$5.449 \cdot 10^{-1}$</td>
<td>$4.742 \cdot 10^{-1}$</td>
<td>$4.393 \cdot 10^{-1}$</td>
</tr>
</tbody>
</table>

Layout
How does length scale influence stresses?

The effect of projection thresholds with $r_{min} 7$:

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$\eta_d = 0.4$</th>
<th>$\eta_d = 0.3$</th>
<th>$\eta_d = 0.2$</th>
<th>$\eta_d = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_e$</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>LS</td>
<td>4.4</td>
<td>6.3</td>
<td>7.7</td>
<td>9.6</td>
</tr>
<tr>
<td>$f^T u$</td>
<td>$2.363 \cdot 10^2$</td>
<td>$2.403 \cdot 10^2$</td>
<td>$2.454 \cdot 10^2$</td>
<td>$2.531 \cdot 10^2$</td>
</tr>
<tr>
<td>$\sigma_{VM}^{max}$</td>
<td>$4.742 \cdot 10^{-1}$</td>
<td>$4.765 \cdot 10^{-1}$</td>
<td>$4.879 \cdot 10^{-1}$</td>
<td>$5.055 \cdot 10^{-1}$</td>
</tr>
</tbody>
</table>

| Layout | | | | |
|--------| | | | |

Stress constraints by length scale control
How does length scale influence stresses?

The effect of filter radius, $\eta_d = 0.4$, $\eta_e = 0.6$:

\[ r_{min} \uparrow \text{compliance} \uparrow \text{stress} \uparrow \]

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<thead>
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<td>4.4</td>
<td>5.7</td>
</tr>
<tr>
<td>$f^T u$</td>
<td>$1.633 \cdot 10^2$</td>
<td>$1.702 \cdot 10^2$</td>
<td>$1.725 \cdot 10^2$</td>
<td>$1.818 \cdot 10^2$</td>
</tr>
<tr>
<td>$\sigma_{VM}^{max}$</td>
<td>$8.186 \cdot 10^{-1}$</td>
<td>$8.307 \cdot 10^{-1}$</td>
<td>$8.388 \cdot 10^{-1}$</td>
<td>$8.591 \cdot 10^{-1}$</td>
</tr>
</tbody>
</table>
The filter radius as a design variable

- The filter radius is treated as a **design variable**
- The maximum stress is treated as a **function of the filter radius**

Minimum compliance optimization in two nested loops:

1. Set initial filter radius, then repeat:
   1. Standard minimum compliance (inner loop)
   2. Evaluate: $d \sigma_{\text{max}} dr_{\text{min}}$
   3. Update: $r_{k+1, \text{min}} = r_{k, \text{min}} + \sigma_{\star, \text{max}} - \sigma_{\text{max}}(r_{k, \text{min}}) dr_{\text{min}}$

Relying on Bendsøe, Diaz and Kikuchi, 1993:

Stress constraints by length scale control
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Relying on Bendsøe, Diaz and Kikuchi, 1993:

$$\frac{1}{E} \frac{2(1+v)}{3} \sigma^T M \sigma \leq \sigma^T C \sigma \leq \frac{1}{E} 2(1-v) \sigma^T M \sigma$$

Stress constraints by length scale control
Adaptive filter radius

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Initial $r_{\text{min}} / \text{LS}$</th>
<th>Final $r_{\text{min}} / \text{LS}$</th>
<th>$\sigma^*_{\text{max}}$</th>
<th>$\sigma^{\text{max}}_{\text{VM}}$</th>
<th>$f^T u$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3.00 / 1.90</td>
<td>5.69 / 3.60</td>
<td>5.00 · 10^{-1}</td>
<td>5.069 · 10^{-1}</td>
<td>2.291 · 10^2</td>
</tr>
<tr>
<td></td>
<td>5.00 / 3.16</td>
<td>8.86 / 5.60</td>
<td>4.500 · 10^{-1}</td>
<td>4.472 · 10^{-1}</td>
<td>2.372 · 10^2</td>
</tr>
<tr>
<td></td>
<td>7.00 / 4.43</td>
<td>11.28 / 7.13</td>
<td>4.000 · 10^{-1}</td>
<td>4.142 · 10^{-1}</td>
<td>2.434 · 10^2</td>
</tr>
</tbody>
</table>

*Layout*

Stress constraints by length scale control
Adaptive filter radius

The adaptive filter radius can give superior combinations of compliance and max. stress:

<table>
<thead>
<tr>
<th>Adaptive $r_{min}$</th>
<th>Constant $r_{min}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f^T u = 2.291 \cdot 10^2$</td>
<td>$f^T u = 2.322 \cdot 10^2$</td>
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<td>$\sigma_{VM}^{max} = 5.069 \cdot 10^{-1}$</td>
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<td>$\sigma_{VM}^{max} = 4.142 \cdot 10^{-1}$</td>
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</table>
Adaptive filter radius

Stress constraints by length scale control
Looking at the required length scale

In some cases, compliance and stress require different length scales in different regions of the design:

Result with $r_{\text{min}} = 3$
low compliance

Result with $r_{\text{min}} = 9$
low max. stress
Looking at the required length scale

In some cases, compliance and stress require different length scales in different regions of the design:

Result with $r_{min} = 3$
- low compliance

Result with $r_{min} = 9$
- low max. stress

**Question**: how can separate length scales be accommodated such that compliance is minimized and stress constraints are satisfied?
A spatially varying filter radius

The length scale (controlled by filter radius) can be seen a spatially varying property:

- Define a critical “stress attractor” point
- Define an auxiliary function:

\[ \phi(x, y) = \exp(-\left| \frac{d(x, y)}{D} \right|^\theta) \quad 0 \leq \phi(x, y) \leq 1 \]

- Parameters: \( D \) is the characteristic influenced distance; \( \theta \) determines the sharpness of \( \phi(x, y) \)
A spatially varying filter radius

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\[
\phi(x, y) = \exp(-\frac{d(x, y)^\theta}{D}) \quad 0 \leq \phi(x, y) \leq 1
\]

- Parameters: \(D\) is the characteristic influenced distance; \(\theta\) determines the sharpness of \(\phi(x, y)\)
- Spatial filter radius is defined as:

\[
\hat{r}_{min}(x, y) = (1 + \gamma\phi(x, y))r_{min}
\]

- Parameters: \(r_{min}\) is the native filter radius; \(\gamma\) is the increase in filter radius at the attractor point
A spatially varying filter radius

Preliminary result with \( r_{min} = 5, \ D = 50, \ \theta = 5, \ \gamma = 2 \):

- **Compliance** \( f^T u = 2.456 \cdot 10^2 \approx 2.445 \cdot 10^2 = f^T u(r_{min} = 9) \)
- **Max. stress** \( \sigma_{VM}^{\text{max}} = 3.153 \cdot 10^{-1} \ll 4.393 \cdot 10^{-1} = \sigma_{VM}^{\text{max}}(r_{min} = 9) \)
Spatially varying and adaptive filter radius

<table>
<thead>
<tr>
<th>Initial / Final $r_{min}$</th>
<th>Final $r_{min}$</th>
<th>$D / \gamma$</th>
<th>$\sigma^*_\text{max}$</th>
<th>$\sigma^\text{max}_{VM}$</th>
<th>$f^T u$</th>
<th>Layout</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.00 / 2.97</td>
<td>3.00 / 4.02</td>
<td>20 / 2</td>
<td>4.000 · 10^{-1}</td>
<td>3.806 · 10^{-1}</td>
<td>2.219 · 10^2</td>
<td><img src="image1" alt="Layout" /></td>
</tr>
<tr>
<td>3.00 / 4.02</td>
<td>3.00 / 4.88</td>
<td>30 / 2</td>
<td>3.500 · 10^{-1}</td>
<td>3.346 · 10^{-1}</td>
<td>2.262 · 10^2</td>
<td><img src="image2" alt="Layout" /></td>
</tr>
<tr>
<td></td>
<td>3.00 / 4.88</td>
<td>30 / 3</td>
<td>3.000 · 10^{-1}</td>
<td>2.933 · 10^{-1}</td>
<td>2.321 · 10^2</td>
<td><img src="image3" alt="Layout" /></td>
</tr>
</tbody>
</table>

Stress constraints by length scale control
Spatially varying *and* adaptive filter radius

A look at the stress distributions:

- $r_{min} = 3.00$
- $r_{min} = 2.97$, $D = 20$, $\gamma = 2$
- $r_{min} = 4.02$, $D = 30$, $\gamma = 2$
- $r_{min} = 4.88$, $D = 30$, $\gamma = 3$
Spatially varying filter on-the-fly

In some cases, location of critical stress concentration is not known
⇒ **Identify and create** spatially varying filter
Spatially varying filter on-the-fly

In some cases, location of critical stress concentration is not known
\[ \Rightarrow \textbf{Identify and create} \text{ spatially varying filter} \]

Minimum compliance optimization in two nested loops:

Set initial filter radius, then repeat:

1. Standard minimum compliance (inner loop)
2. Find geometric locations of max. stress violations
3. Sort by stress magnitude
4. Remove duplicates / overlapping regions
5. Generate (a limited number of) auxiliary functions \( \phi_i(x, y) \)
Final example: U-bracket

Constant $r_{\text{min}} = 3.00$

\[
\sigma_{VM}^{\max} = 5.078 \cdot 10^{-1}\quad f^T u = 1.061 \cdot 10^2
\]

Adaptive $r_{\text{min}} = 3.00 \rightarrow 8.00$

\[
\sigma_{VM}^{\max} = 4.125 \cdot 10^{-1}\quad f^T u = 1.291 \cdot 10^2
\]

Spatial $r_{\text{min}} = 3.00 \rightarrow 2.78$, $D = 20$, $\gamma = 2$

\[
\sigma_{VM}^{\max} = 3.862 \cdot 10^{-1}\quad f^T u = 1.177 \cdot 10^2
\]

Spatial automatic $r_{\text{min}} = 3.00$, $D = 20$, $\gamma = 2$

\[
\sigma_{VM}^{\max} = 3.509 \cdot 10^{-1}\quad f^T u = 1.174 \cdot 10^2
\]
Summary & conclusions

- Two approaches for satisfying stress constraints by minimizing compliance with control on length scale:
  - Filter radius is a design variable, determined according to stress constraint
  - Filter radius varies spatially, according to stress level
- For smooth stress distributions, stresses are minimized together with compliance
- Promising results – reduction in maximum stresses
Summary & conclusions

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QUESTIONS???
Extras

- It may not be necessary to symmetrize the filter operator:

\[ f^T u = 2.460 \cdot 10^2 \quad \sigma_{VM}^{\text{max}} = 3.133 \cdot 10^{-1} \]
Extras

- Area and curvature constraints using B-splines: