An alternative approach for satisfying stress constraints in continuum topology optimization using nonlinear material modeling

Oded Amir
odedamir@cv.technion.ac.il

Technion - Israel Institute of Technology
Faculty of Civil & Environmental Engineering

WCSMO-11, 9 June 2015, Sydney, AUSTRALIA
The challenges of stress constraints

Characteristics of stress-constrained continuum topology optimization:

- Basic engineering requirement: remain linear-elastic
- Local measure $\rightarrow$ large number of constraints
- Removal of material $\rightarrow$ vanishing of constraint

Challenge #1: COMPLEXITY
Large number of design variables, large number of constraints

Challenge #2: SINGULARITY
Difficult to capture true optimum by numerical procedures
The challenges of stress constraints

Characteristics of stress-constrained continuum topology optimization:
- Basic engineering requirement: remain linear-elastic
- Local measure $\rightarrow$ large number of constraints
- Removal of material $\rightarrow$ vanishing of constraint

Challenge #1: COMPLEXITY
Large number of design variables, large number of constraints

Challenge #2: SINGULARITY
Difficult to capture true optimum by numerical procedures

Alternative approach for satisfying stress constraints
Dealing with the challenge of complexity (1/2)

Strategy 1: consider all local constraints, solve with “active” subsets
[Duysinx and Bendsøe, 1998], [Bruggi and Duysinx, 2012]
[Pereira et al., 2004], [Fancello, 2006] - Augmented Lagrangian

Strategy 2: aggregate local constraints into global stress function, using K-S or p-norm functions
[Yang and Chen, 1996], [Park, 1995], [Duysinx and Sigmund, 1998]
[París et al., 2007], [Le et al., 2010], [París et al., 2010] - regional block aggregation

Other approaches
[Amstutz and Novotny, 2010] - topological derivative, external penalty
[Verbart et al., 2013] - artificial damage
Dealing with the challenge of complexity (1/2)

**Strategy 1: consider all local constraints, solve with “active” subsets**

[Duysinx and Bendsøe, 1998], [Bruggi and Duysinx, 2012]
[Pereira et al., 2004], [Fancello, 2006] - Augmented Lagrangian

**Strategy 2: aggregate local constraints into global stress function, using K-S or p-norm functions**

[Yang and Chen, 1996], [Park, 1995], [Duysinx and Sigmund, 1998]
[París et al., 2007], [Le et al., 2010], [París et al., 2010] - regional block aggregation

**Other approaches**

[Amstutz and Novotny, 2010] - topological derivative, external penalty
[Verbart et al., 2013] - artificial damage

Alternative approach for satisfying stress constraints
Dealing with the challenge of complexity (1/2)

Strategy 1: consider all local constraints, solve with “active” subsets
[Duysinx and Bendsøe, 1998], [Bruggi and Duysinx, 2012]
[Pereira et al., 2004], [Fancello, 2006] - Augmented Lagrangian

Strategy 2: aggregate local constraints into global stress function, using K-S or p-norm functions
[Yang and Chen, 1996], [Park, 1995], [Duysinx and Sigmund, 1998]
[París et al., 2007], [Le et al., 2010], [París et al., 2010] - regional block aggregation

Other approaches
[Amstutz and Novotny, 2010] - topological derivative, external penalty
[Verbart et al., 2013] - artificial damage
Dealing with the challenge of complexity (1/2)

**Strategy 1: consider all local constraints, solve with “active” subsets**

[Duysinx and Bendsøe, 1998], [Bruggi and Duysinx, 2012]  
[Pereira et al., 2004], [Fanceollo, 2006] - Augmented Lagrangian

**Strategy 2: aggregate local constraints into global stress function, using K-S or p-norm functions**

[Yang and Chen, 1996], [Park, 1995], [Duysinx and Sigmund, 1998]  
[París et al., 2007], [Le et al., 2010], [París et al., 2010] - regional block aggregation

**Other approaches**

[Amstutz and Novotny, 2010] - topological derivative, external penalty  
[Verbart et al., 2013] - artificial damage
Dealing with the challenge of complexity (2/2)

![Image of U-beam: boundary conditions, obtained design and zoom near a reentrant corner](image1)

![Image of L-beam](image2)

Table 4 L-beam

<table>
<thead>
<tr>
<th>L-beam</th>
<th>αβ</th>
<th>¯σM</th>
<th>Area</th>
<th>Compliance</th>
<th>max</th>
<th>Ω σM(uΩ)</th>
<th>CPU time (s)</th>
<th>Mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>104</td>
<td>10</td>
<td>−2</td>
<td>40</td>
<td>0.56</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>−2</td>
<td>50</td>
<td>0.65</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>−2</td>
<td>60</td>
<td>0.75</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>10</td>
<td>−2</td>
<td>70</td>
<td>0.85</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: The full domain Ω₀ = D.

Fig. 10 U-beam: boundary conditions, obtained design and zoom near a reentrant corner

Alternative approach for satisfying stress constraints

Dealing with the challenge of complexity (2/2)

We observe any adverse effects as the designs converge.

We also note that the proposed evolving region definition further adds to the potentially problematic definition differs. We also note that the proposed evolving that the regional constraint is similar to the “block aggregation” approaches.

For the values of m in which the admissible stress criterion is imposed as a hard constraint, not applying stress relaxation in which the admissible stress criterion is imposed as a hard constraint, not applying stress relaxation is effectively removed and the stress distribution becomes meaningful stress-based design. The main reason to use a relaxed stress, as we did in most problems, is to get meaningful stress-based results.

The admissible stress criterion is not imposed as a hard stress constraint and therefore there are no vanishing stress measures using aggregation functions in order to reduce computational costs related to the sensitivity computation we are able to significantly improve the respect.

We improve. However, in practice the optimizer may converge to a non-optimal solution which is a drawback of the method. To overcome this, we propose an alternative approach for satisfying stress constraints by means of a sequence of approximations. The method is based on the fact that the stress distribution at any point in the design domain is the average of the stresses at the corresponding points in the reference domain. The sequence of approximations is obtained by iteratively refining the design domain and updating the stress distribution accordingly.

The main advantages of this method are:

1. It can handle complex stress distributions and constraints.
2. It allows for a smooth transition between the relaxed and unrelaxed stress definitions.
3. It can be used with any optimizer that is capable of handling relaxed stress constraints.

The main disadvantages are:

1. It may not always converge to the optimal solution.
2. It can be computationally expensive if the design domain is large or if the stress distribution is complex.

In summary, the method presented in this section provides an alternative approach for satisfying stress constraints in topology optimization problems. It offers a flexible and robust framework that can be adapted to different optimization problems and design domains.
Alternative approach based on elasto-plasticity (1/2)

Find a stress-constrained layout by modeling post-yielding response and driving the design towards “no-yield”

i.e.

Minimize plastic strains s.t.
volume and compliance
Alternative approach based on elasto-plasticity (1/2)

Find a stress-constrained layout by modeling post-yielding response and driving the design towards “no-yield”

i.e.

Minimize plastic strains s.t.
volume and compliance

(a) Step 1: 36 design iterations with $p_E = 1.5, p_{\sigma_y} = 1.0$, filter radius $r = 0.015$ and hardening $H = 0.001$.

(b) Step 2: 35 further design iterations with $p_E = 2.0, p_{\sigma_y} = 1.5$, filter radius $r = 0.015$ and hardening $H = 0.001$.

(c) Step 3: 51 further design iterations with $p_E = 2.5, p_{\sigma_y} = 2.0$, filter radius $r = 0.015$ and hardening $H = 0.001$.

(d) Step 4: 35 further design iterations with $p_E = 3.0, p_{\sigma_y} = 2.5$, filter radius $r = 0.010$ and hardening $H = 0.01$.

[Amir, 2011]
Alternative approach based on elasto-plasticity (2/2)

**Current work:**

Minimize volume s.t. plastic strains ($= 0$) and end-compliance

Key aspects:

- Stress constraints are evaluated **accurately** at local material points;
- Formulation involves constraints on **global** quantities only;
- **Nonlinear FE** analysis is required.

Relative complexity: ↑ analysis ↓ optimization
Alternative approach based on elasto-plasticity (2/2)

Current work:

Minimize volume s.t. plastic strains \( (\varepsilon = 0) \) and end-compliance

Key aspects:

- Stress constraints are evaluated **accurately** at local material points;
- Formulation involves constraints on **global** quantities only;
- **Nonlinear FE** analysis is required.

Relative complexity: ↑ analysis ↓ optimization
Alternative approach based on elasto-plasticity (2/2)

Current work:
Minimize volume s.t. plastic strains ($= 0$) and end-compliance

Key aspects:
- Stress constraints are evaluated accurately at local material points;
- Formulation involves constraints on global quantities only;
- **Nonlinear FE** analysis is required.

Relative complexity: ↑ analysis ↓ optimization
Topology optimization with elasto-plasticity

Optimizing the energy absorbed by the structure
[Yuge and Kikuchi, 1995], [Swan and Kosaka, 1997], [Yuge et al., 1999],
[Maute et al., 1998], [Schwarz et al., 2001], [Yoon and Kim, 2007],
[Kato et al., 2015]

Crashworthiness design
e.g. [Pedersen, 2004]

Concrete / steel layouts
[Bogomonly and Amir, 2012]

Effective energy dissipation
[Nakshatrala and Tortorelli, 2015]

Failure mitigation based on continuum damage modeling
[James and Waisman, 2014]
Topology optimization with elasto-plasticity

Optimizing the energy absorbed by the structure
[Yuge and Kikuchi, 1995], [Swan and Kosaka, 1997], [Yuge et al., 1999],
[Maute et al., 1998], [Schwarz et al., 2001], [Yoon and Kim, 2007],
[Kato et al., 2015]

Crashworthiness design
E.g. [Pedersen, 2004]

Concrete / steel layouts
[Bogomonly and Amir, 2012]

Effective energy dissipation
[Nakshatrala and Tortorelli, 2015]

Failure mitigation based on continuum damage modeling
[James and Waisman, 2014]

Alternative approach for satisfying stress constraints
Topology optimization with elasto-plasticity

Optimizing the energy absorbed by the structure
[Yuge and Kikuchi, 1995], [Swan and Kosaka, 1997], [Yuge et al., 1999], [Maute et al., 1998], [Schwarz et al., 2001], [Yoon and Kim, 2007], [Kato et al., 2015]

Crashworthiness design
e.g. [Pedersen, 2004]

Concrete / steel layouts
[Bogomonly and Amir, 2012]

Effective energy dissipation
[Nakshatrala and Tortorelli, 2015]

Failure mitigation based on continuum damage modeling
[James and Waisman, 2014]
Topology optimization with elasto-plasticity

Optimizing the energy absorbed by the structure
[Yuge and Kikuchi, 1995], [Swan and Kosaka, 1997], [Yuge et al., 1999],
[Maute et al., 1998], [Schwarz et al., 2001], [Yoon and Kim, 2007],
[Kato et al., 2015]

Crashworthiness design
e.g. [Pedersen, 2004]

Concrete / steel layouts
[Bogomonly and Amir, 2012]

Effective energy dissipation
[Nakshatrala and Tortorelli, 2015]

Failure mitigation based on continuum damage modeling
[James and Waisman, 2014]
Topology optimization with elasto-plasticity

Optimizing the energy absorbed by the structure
[Yuge and Kikuchi, 1995], [Swan and Kosaka, 1997], [Yuge et al., 1999],
[Maute et al., 1998], [Schwarz et al., 2001], [Yoon and Kim, 2007],
[Kato et al., 2015]

Crashworthiness design
e.g. [Pedersen, 2004]

Concrete / steel layouts
[Bogomonly and Amir, 2012]

Effective energy dissipation
[Nakshatrala and Tortorelli, 2015]

Failure mitigation based on continuum damage modeling
[James and Waisman, 2014]
Topology optimization with elasto-plasticity

Optimizing the energy absorbed by the structure
[Yuge and Kikuchi, 1995], [Swan and Kosaka, 1997], [Yuge et al., 1999],
[Maute et al., 1998], [Schwarz et al., 2001], [Yoon and Kim, 2007],
[Kato et al., 2015]

Crashworthiness design
e.g. [Pedersen, 2004]

Concrete / steel layouts
[Bogomonly and Amir, 2012]

Effective energy dissipation
[Nakshatrala and Tortorelli, 2015]

Failure mitigation based on continuum damage modeling
[James and Waisman, 2014]
Governing equations - elasto-plasticity

Rate-independent plasticity, $J_2$ flow theory:

von Mises yield criterion:
$$ f(\sigma, \kappa) = \sqrt{3J_2} - \sigma_y(\kappa) \leq 0 $$

Bi-linear isotropic hardening:
$$ \sigma_y(\kappa) = \sigma_y^0 + HE\kappa $$

Associative flow rule:
$$ \dot{\epsilon}^{pl} = \dot{\lambda} \frac{\partial f}{\partial \sigma} $$

Evolution of internal hardening variable:
$$ \dot{\kappa} = \sqrt{\frac{2}{3}} \left\| \dot{\epsilon}^{pl} \right\|_2 $$

Solution on a local level by well-known return mapping algorithm
[Simo and Taylor, 1986].

Alternative approach for satisfying stress constraints
Nonlinear FEA

Recasting as a nonlinear, transient coupled problem

[Michaleris et al., 1994]:

\[ n \mathbf{R}(u^n, u^{n-1}, v^n, v^{n-1}) = 0 \]
\[ n \mathbf{H}(u^n, u^{n-1}, v^n, v^{n-1}) = 0 \]

\[ n \mathbf{v} = \begin{bmatrix} n \varepsilon_{pl} \\ n \kappa \\ n \sigma \\ n \lambda \end{bmatrix} \]

Global incremental force equilibrium, displacement control:

\[ n \mathbf{R}(v^n, \theta^n) = n \dot{\theta} \mathbf{f}_{ext} - \int_V \mathbf{B}^T n \mathbf{\sigma} dV \]

Local incremental constitutive equations:

\[ n \mathbf{H}_1 = n^{-1} \varepsilon_{pl} + (n \lambda - n^{-1} \lambda)(\frac{\partial f}{\partial n \sigma})^T - n \varepsilon_{pl} \quad \text{(associative flow)} \]
\[ n \mathbf{H}_2 = n^{-1} \kappa + (n \lambda - n^{-1} \lambda)\sqrt{\frac{2}{3}}(\frac{\partial f}{\partial n \sigma})^T(\frac{\partial f}{\partial n \sigma}) - n \kappa \quad \text{(hardening variable)} \]
\[ n \mathbf{H}_3 = n^{-1} \sigma + \mathbf{D} \left[ \mathbf{B}^n \mathbf{u} - \mathbf{B}^{n-1} \mathbf{u} - (n \varepsilon_{pl} - n^{-1} \varepsilon_{pl}) \right] - n \mathbf{\sigma} \quad \text{(elastic stress-strain)} \]
\[ n \mathbf{H}_4 = J_2 - \frac{1}{3}(\sigma_y(\kappa))^2 \quad \text{(yield surface)} \]

Alternative approach for satisfying stress constraints
Nonlinear FEA

Recasting as a nonlinear, transient coupled problem

\[ nR(n_u, n_{-1} u, n_v, n_{-1} v) = 0 \]
\[ nH(n_u, n_{-1} u, n_v, n_{-1} v) = 0 \]

\[ n_v = \begin{bmatrix} n\epsilon_{pl} \\ n\kappa \\ n\sigma \\ n\lambda \end{bmatrix} \]

Global incremental force equilibrium, displacement control:

\[ nR(n_v, n_\theta) = n_\theta f_{ext} - \int_V B^T n_\sigma dV \]

Local incremental constitutive equations:

\[ nH_1 = n_{-1} \epsilon_{pl} + (n_\lambda - n_{-1} \lambda) \left( \frac{\partial f}{\partial n_\sigma} \right)^T - n\epsilon_{pl} \quad \text{(associative flow)} \]
\[ nH_2 = n_{-1} \kappa + (n_\lambda - n_{-1} \lambda) \sqrt{\frac{2}{3} \left( \frac{\partial f}{\partial n_\sigma} \right)^T \frac{\partial f}{\partial n_\sigma}} - n\kappa \quad \text{(hardening variable)} \]
\[ nH_3 = n_{-1} \sigma + D \left[ B^n u - B^{n-1} u - (n\epsilon_{pl} - n_{-1} \epsilon_{pl}) \right] - n\sigma \quad \text{(elastic stress-strain)} \]
\[ nH_4 = J_2 - \frac{1}{3} (\sigma_y(\kappa))^2 \quad \text{(yield surface)} \]

Alternative approach for satisfying stress constraints
Problem formulation

\[
\min_{x} f(x) = \frac{\sum_{i=1}^{N_{\text{elem}}} v_i \bar{x}_i}{N_{\text{elem}}} \quad \text{(volume fraction)}
\]

s.t.:  
\[
g_1(x) = -N \hat{\theta} f_{\text{ext}}^T N u + g^* \leq 0 \quad \text{(end-compliance, disp. control)}
\]

\[
g_2(x) = \sum_{i=1}^{N_{\text{elem}}} \sum_{j=1}^{N_{\text{gpts}}} N \kappa_{i,j} \leq 0 \quad \text{(plastic strains)}
\]

\[
0 \leq x_i \leq 1, \quad i = 1, \ldots, N_{dv}
\]

with:

\[
R_n(\nu, \theta) = 0 \quad n = 1, \ldots, N
\]

\[
H_n(u, \nu, \nu, \bar{x}) = 0 \quad n = 1, \ldots, N
\]

Remarks:

- Physical density \( \bar{x} \) from density filter and Heaviside projection  
  [Bruns and Tortorelli, 2001, Bourdin, 2001, Guest et al., 2004, Xu et al., 2010].

- Solution obtained by MMA [Svanberg, 1987].

Alternative approach for satisfying stress constraints
Design parameterization

Modified SIMP for stiffness and yield stress

[Bendsøe, 1989, Sigmund and Torquato, 1997]:

\[ E(\bar{x}_i) = E_{\text{min}} + (E_{\text{max}} - E_{\text{min}})\bar{x}_i^{p_E} \]

\[ \sigma^0_y(\bar{x}_i) = \sigma^0_{y,\text{min}} + (\sigma^0_{y,\text{max}} - \sigma^0_{y,\text{min}})\bar{x}_i^{p_{\sigma_y}} \]

\[ p_E > p_{\sigma_y} \quad [\text{Maute et al., 1998}] \]

↓

“delayed” yield strain for intermediate densities

↓

relaxation of singularity

Alternative approach for satisfying stress constraints
Adjoint sensitivity analysis

Backwards-incremental adjoint procedure [Michaleris et al., 1994]:

Augmented response functional

\[ \hat{g}(\mathbf{u}, \mathbf{v}, \theta, \bar{x}) = g(Nu, Nv, N\theta, \bar{x}) - \sum_{n=1}^{N} n^n R(n^v, n^\theta) - \sum_{n=1}^{N} n^n H(nu, n^{-1}u, n^v, n^{-1}v, \bar{x}) \]

Global adjoint equations for \( n^\lambda \)

\[
\begin{align*}
\left[-\frac{\partial(n^R)}{\partial(n^v)} \frac{\partial(n^H)}{\partial(n^v)} - 1 \frac{\partial(n^H)}{\partial(n^u)} \right]^T n^\lambda &= \frac{\partial g}{\partial(n^u)} - \left[ \frac{\partial g}{\partial(n^v)} \frac{\partial(n^H)}{\partial(n^v)} - 1 \frac{\partial(n^H)}{\partial(n^u)} \right]^T \\
&\quad\quad - \left[ \frac{\partial(n+1^H)}{\partial(n^u)} - \frac{\partial(n+1^H)}{\partial(n^v)} \frac{\partial(n^H)}{\partial(n^v)} - 1 \frac{\partial(n^H)}{\partial(n^u)} \right]^T n^+1^\gamma \\
\frac{\partial(n^R)}{\partial(n^\theta)} n^\lambda &= \frac{\partial g}{\partial(n^\theta)}
\end{align*}
\]

Local adjoint equations for \( n^\gamma \)

\[
\frac{\partial(n^H)}{\partial(n^v)} n^\gamma = - \frac{\partial(n^R)}{\partial(n^v)} n^\lambda - \frac{\partial(n+1^H)}{\partial(n^v)} n^+1^\gamma + \frac{\partial g}{\partial(n^v)}^T
\]

Explicit derivatives w.r.t. design variables

\[
\frac{\partial \hat{g}_{exp}}{\partial \tilde{x}_i} = \frac{\partial g}{\partial \tilde{x}_i} - \sum_{n=1}^{N} n^\gamma T \frac{\partial n^H}{\partial \tilde{x}_i}
\]

Alternative approach for satisfying stress constraints
Example: L-bracket (1/3)

Solution parameters:

- $n_{elx} = n_{ely} = 160$, filter radius = 0.025
- $\delta = 0.01$
- $E_{min} = 0.001$, $E_{max} = 1000$, $\sigma_{y,min}^0 = 0$, $\sigma_{y,max}^0 = 1.8$
- Continuation on $p_E$, $p_{\sigma_y}$, $\beta$

Alternative approach for satisfying stress constraints
Example: L-bracket (2/3)
Example: L-bracket (3/3)

In the optimized design:

- Reentrant corner is circumvented;
- Compliance constraint is satisfied;
- Maximum stress is the allowable stress.
Summary and conclusions

- **Stress constraints** can be achieved via elasto-plastic modeling by minimizing or constraining the sum of plastic strains;
- Stress violations are captured accurately without local constraints;
- Computational cost dominated by NLFEA - can be competitive in large scale;
- Oscillatory behavior - still much room for improvements.

Alternative approach for satisfying stress constraints
Work in progress - oscillations

Volume minimization

Alternative approach for satisfying stress constraints
Stress

Alternative approach for satisfying stress constraints
Sensitivity of stress

\[ pE = 3, \ pS = 2.5, \ H = 0.01 \]

- Normalized strain
- Derivative of stress

Alternative approach for satisfying stress constraints
Strain energy vs. strain

\[ pE = 3, \ pS = 2.5, \ H = 0.01 \]

Normalized strain

Strain energy

Alternative approach for satisfying stress constraints
Sensitivity of strain energy vs. strain

$pE = 3, pS = 2.5, H = 0.01$

Derivative of strain energy vs. normalized strain

Alternative approach for satisfying stress constraints
Strain energy vs. stress

$pE = 3, pS = 2.5, H = 0.01$

Alternative approach for satisfying stress constraints
Sensitivity of strain energy vs. stress

pE = 3, pS = 2.5, H = 0.01

Derivative of strain energy vs. normalized stress

Alternative approach for satisfying stress constraints
References I


References II

Topology optimization of non-linear elastic structures and compliant mechanisms.

Topology optimization of continuum structures with local stress constraints.

New developments in handling stress constraints in optimal material distribution.

Topology optimization for minimum mass design considering local failure constraints and contact boundary conditions.

Achieving minimum length scale in topology optimization using nodal design variables and projection functions.

Failure mitigation in optimal topology design using a coupled nonlinear continuum damage model.
Analytical sensitivity in topology optimization for elastoplastic composites.

Stress-based topology optimization for continua.

Adaptive topology optimization of elastoplastic structures.

Tangent operators and design sensitivity formulations for transient non-linear coupled problems with applications to elastoplasticity.

Topology optimization for effective energy propagation in rate-independent elastoplastic material systems.
*Computer Methods in Applied Mechanics and Engineering*.

Block aggregation of stress constraints in topology optimization of structures.

Alternative approach for satisfying stress constraints
References IV


*Extensions of optimal layout design using the homogenization method.* PhD thesis, University of Michigan, Ann Arbor.


The method of moving asymptotes - a new method for structural optimization. 

Voigt-reuss topology optimization for structures with nonlinear material behaviors. 

In *10th World Congress on Structural and Multidisciplinary Optimization*, Orlando, Florida, USA.

Volume preserving nonlinear density filter based on heaviside functions. 

Stress-based topology optimization. 

Topology optimization of material-nonlinear continuum structures by the element connectivity parameterization. 

Optimization of 2-d structures subjected to nonlinear deformations using the homogenization method. 
*Structural optimization*, 17(4):286–299.

**Alternative approach for satisfying stress constraints**