Efficient Use of Iterative Solvers in Nested Topology Optimization

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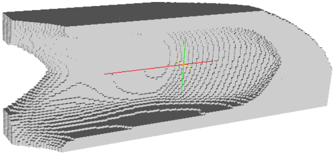


Introduction - what is this presentation about?

- **Nested approach** to structural optimization, with focus on **topology optimization**.
- Iterative solution of the nested equation system using **Krylov subspace solvers**.
- Case studies are from the field of structural mechanics (linear elasticity), so we focus on the use of **Preconditioned Conjugate Gradients** as the iterative solver.

Main theme

Using approximate solutions in the analysis problem can save significant computing time, without affecting the accuracy of the optimization significantly .

Introduction - the nested approach

The nested approach to topology optimization, demonstrated on a **minimum compliance** problem:

$$\min_{\boldsymbol{\rho}} c(\boldsymbol{\rho}) = \mathbf{f}^T\mathbf{u} \qquad \text{(compliance)}$$

$$\text{s.t.:} \quad \sum_{e=1}^{N} v_e \rho_e \leq V \qquad \text{(volume constraint)}$$

$$0 < \rho_{min} \leq \rho_e \leq 1 \qquad \text{(element densities)}$$

$$\text{with:} \quad \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f} \qquad \text{(equilibrium)}$$

* The main computational bottleneck - solving the (typically large) system of equations $\mathbf{K}\mathbf{u} = \mathbf{f}$ (and additional adjoint systems in other problems).
* Other structural optimization problems may share the same formulation.

Introduction - motivation for this study

In practice, when solving large problems, most of us turn to the iterative family of **Krylov subspace solvers**, e.g. the Conjugate Gradients method with effective preconditioning (PCG):

- Low memory requirements.
- Suitable for parallel computing.

The challenge:
Krylov subspace solvers typically require a **large number of iterations** in order to converge to an accurate solution of the nested problem. Then this should be **repeated for every design iteration**.

Proposed approximation

The common convergence criterion for PCG (and similar methods):

$$\frac{\|\mathbf{f} - \mathbf{K}\mathbf{u}_k\|_2}{\|\mathbf{f}\|_2} = \frac{\|\mathbf{r}_k\|_2}{\|\mathbf{f}\|_2} < \epsilon$$

* A typical value of the tolerance ϵ is 10^{-6}.
* $\mathbf{u}_k$ is in practice, the **accurate** solution.

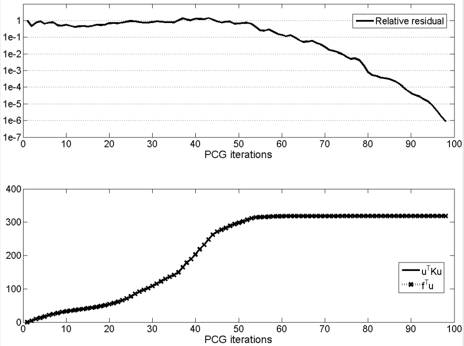
The proposed approximation is $\mathbf{u}_m$, $m < k$

$$\frac{\|\mathbf{f} - \mathbf{K}\mathbf{u}_m\|_2}{\|\mathbf{f}\|_2} = \frac{\|\mathbf{r}_m\|_2}{\|\mathbf{f}\|_2} >> \epsilon$$

* How should the PCG cycle m be chosen?
* Is it enough to use a slack ϵ?

PCG performance - minimum compliance (1)

Relative residual, $\mathbf{f}^T\mathbf{u}$ and $\mathbf{u}^T\mathbf{K}\mathbf{u}$

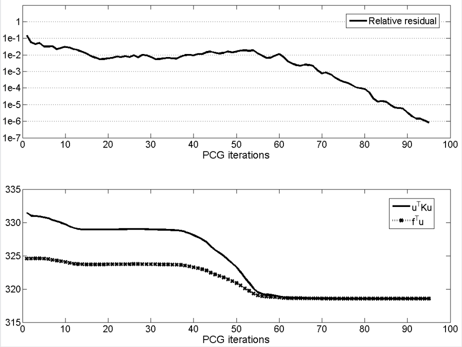


Remarks

* Initial guess $= \mathbf{0}$.
* $\mathbf{f}^T\mathbf{u}_i = \mathbf{u}_i^T\mathbf{K}\mathbf{u}_i$ for all PCG iterations i.
* $\frac{\partial c}{\partial \rho_e}_i = -\mathbf{u}_i^T \frac{\partial \mathbf{K}}{\partial \rho_e}\mathbf{u}_i$.

PCG performance - minimum compliance (2)

Relative residual, $\mathbf{f}^T\mathbf{u}$ and $\mathbf{u}^T\mathbf{K}\mathbf{u}$



Remarks

* Initial guess $= \mathbf{u}(\rho_{old})$.
* In general: $\mathbf{f}^T\mathbf{u}_i \neq \mathbf{u}_i^T\mathbf{K}\mathbf{u}_i$.

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Alternative convergence criteria

Question: How to choose the approximation $\mathbf{u}_m$ so that the objective ($\mathbf{f}^T\mathbf{u}_m$) and sensitivities ($-\mathbf{u}_m^T \frac{\partial \mathbf{K}}{\partial \rho_e} \mathbf{u}_m$) are **sufficiently accurate**?

Initial guess $= \mathbf{0}$

Measure the relative change in $\mathbf{K}$-norm of the solution:

$$\frac{\mathbf{u}_m^T \mathbf{K} \mathbf{u}_m - \mathbf{u}_{m-1}^T \mathbf{K} \mathbf{u}_{m-1}}{\mathbf{u}_{m-1}^T \mathbf{K} \mathbf{u}_{m-1}} < \epsilon$$

Initial guess $= \mathbf{u}(\rho_{old})$

Measure the relative difference between compliance and $\mathbf{K}$-norm of the solution

$$\frac{\left|\mathbf{f}^T \mathbf{u}_m - \mathbf{u}_m^T \mathbf{K} \mathbf{u}_m\right|}{\mathbf{u}_m^T \mathbf{K} \mathbf{u}_m} < \epsilon$$

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The force inverter problem

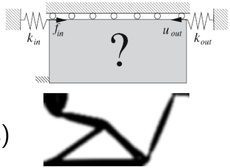
The nested approach to topology optimization, demonstrated on a **force inverter** problem:

$$\min_{\rho} c(\rho) = \mathbf{l}^T \mathbf{u} \qquad \text{(output)}$$

$$\text{s.t.:} \quad \sum_{e=1}^{N} v_e \rho_e \leq V \qquad \text{(volume constraint)}$$

$$0 < \rho_{min} \leq \rho_e \leq 1, \qquad \text{(element densities)}$$

$$\text{with:} \quad \mathbf{K}(\rho)\mathbf{u} = \mathbf{f} \qquad \text{(equilibrium)}$$



* The aim is to maximize an output displacement, designated by the vector $\mathbf{l}$.
* Sensitivity analysis requires the solution of an adjoint system: $\mathbf{K}(\rho)^T \boldsymbol{\lambda} = \mathbf{l}$.
* $\frac{\partial c}{\partial \rho_e} = -\boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial \rho_e} \mathbf{u}$.

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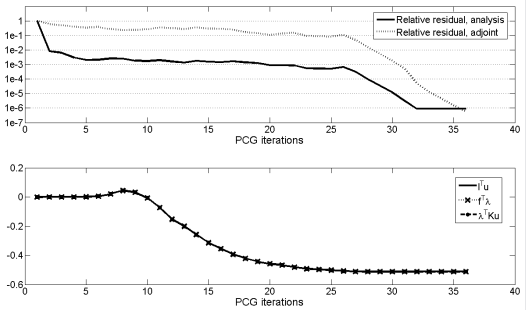
PCG performance - force inverter (1)

Relative residual, $\mathbf{l}^T\mathbf{u}$, $\mathbf{f}^T\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^T\mathbf{K}\mathbf{u}$



Remarks

* Both r.h.s. solved together by block-PCG.
* Initial guess $= \mathbf{0}$.
* $\mathbf{l}^T\mathbf{u}_i = \mathbf{f}^T\boldsymbol{\lambda}_i = \boldsymbol{\lambda}_i^T\mathbf{K}\mathbf{u}_i$ for all PCG iterations i.

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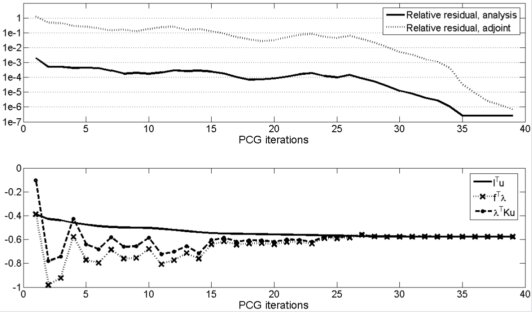
PCG performance - force inverter (2)

Relative residual, $\mathbf{l}^T\mathbf{u}$, $\mathbf{f}^T\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^T\mathbf{K}\mathbf{u}$



Remarks

* Initial guess $= \mathbf{u}(\rho_{old}), \boldsymbol{\lambda}(\rho_{old})$.
* In general: $\mathbf{l}^T\mathbf{u}_i \neq \boldsymbol{\lambda}_i^T\mathbf{K}\mathbf{u}_i$.
* In general: $\mathbf{f}^T\boldsymbol{\lambda}_i \neq \boldsymbol{\lambda}_i^T\mathbf{K}\mathbf{u}_i$.

Alternative convergence criteria

Question: How to choose the approximations $\mathbf{u}_m$ and $\boldsymbol{\lambda}_m$ so that the objective ($\mathbf{l}^T\mathbf{u}_m$) and sensitivities ($-\boldsymbol{\lambda}_m^T \frac{\partial \mathbf{K}}{\partial \rho_e}\mathbf{u}_m$) are **sufficiently accurate**?

Initial guess $= \mathbf{0}$

Measure the relative change in the value of the objective:

$$\left|\frac{\mathbf{l}^T\mathbf{u}_m - \mathbf{l}^T\mathbf{u}_{m-1}}{\mathbf{l}^T\mathbf{u}_{m-1}}\right| < \epsilon$$

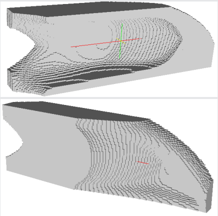
Initial guess $= \mathbf{u}(\boldsymbol{\rho}_{old}), \boldsymbol{\lambda}(\boldsymbol{\rho}_{old})$

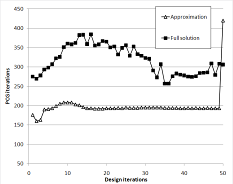
Measure **also** the relative difference between the objective and $\mathbf{f}^T\boldsymbol{\lambda}_i$:

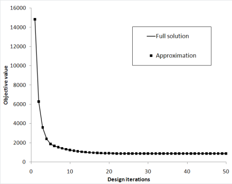
$$\left|\frac{\mathbf{l}^T\mathbf{u}_m - \mathbf{f}^T\boldsymbol{\lambda}_m}{\mathbf{f}^T\boldsymbol{\lambda}_m}\right| < \epsilon$$

Example: large scale minimum compliance

* 324,000 elements, 1.03E6 DOF.
* Preconditioned with IC(0).
* $\approx$ 40% reduction in PCG iterations.
* After 50 design iterations: 0.02% error in objective value.
* After 50 design iterations: Same topology, some differences in boundary regions.







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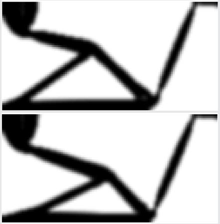
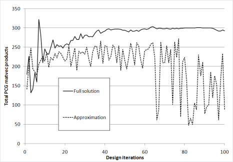
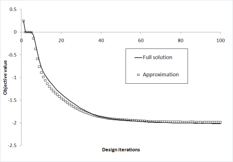
Example: force inverter

- 7,200 elements, 14,762 DOF.
- Preconditioned with IC(0).
- $\approx$ 30% reduction in PCG iterations.
- After 100 design iterations: 1.5% error in objective value.
- After 100 design iterations: Same topology but different shape.







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Concluding remarks

- Achieved savings: $\approx 40\%$ in 3D minimum compliance, $\approx 30\%$ in 2D force inverter.
- Improving accuracy of force inverter problems and extending to 3D are among future tasks.
- Further interesting extensions:
  - Other physical models, objective functions.
  - Other Krylov solvers besides PCG.
- **The key point**: convergence measures should be related to the objective function and to the corresponding design sensitivities.