On Multigrid-CG for Efficient Topology Optimization

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Increasing interest in topology optimization within the architectural community $\rightarrow$ development of plug-ins and add-ons to CAD software.

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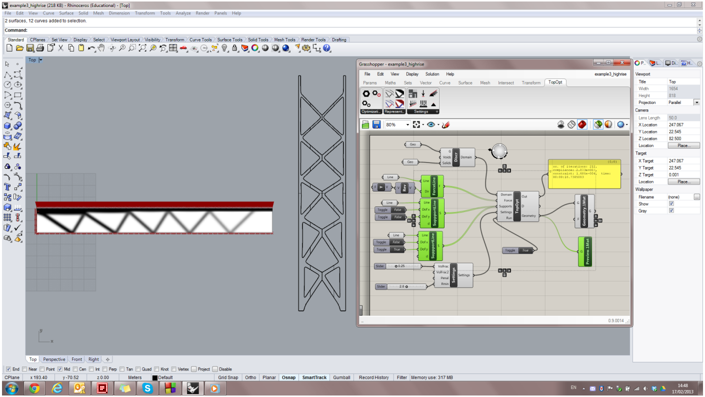

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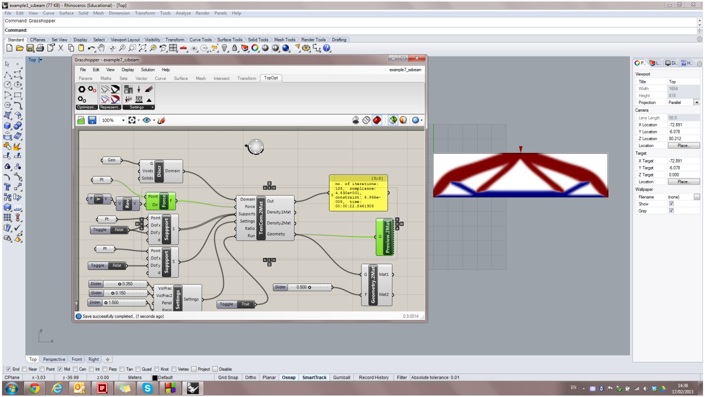

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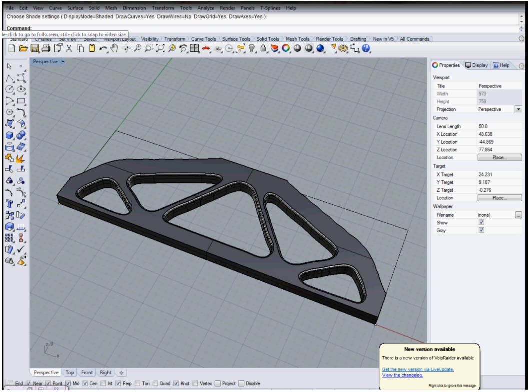

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- **3-D still very challenging on standard computers - main aim of current study.**
- Insight also relevant for high-performance parallel environments.

Nested formulation

$$
\begin{aligned}
\min_{\boldsymbol{\rho}} \phi(\boldsymbol{\rho}) \quad &= \quad \mathbf{l}^{\mathsf{T}}\mathbf{u} \\
\text{s.t.:} \quad & \quad \sum_{e=1}^{N} v_e \rho_e \leq V \\
& \quad 0 \leq \rho_e \leq 1 \qquad e = 1, \ldots, N \\
\text{with:} \quad & \quad \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f}
\end{aligned}
$$

- Modified SIMP $E(\rho) = E_{min} + (E_{max} - E_{min})\rho^p$.
- Sensitivities are $\frac{\partial \phi}{\partial \rho_e} = -\overline{\mathbf{u}}^{\mathsf{T}} \frac{\partial \mathbf{K}}{\partial \rho_e}\mathbf{u}$, with $\mathbf{K}(\boldsymbol{\rho})\overline{\mathbf{u}} = \mathbf{l}$.
- Solution obtained by optimality criteria or nonlinear programming.
- Computational cost dominated by nested equations: $\mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f}$, $\mathbf{K}(\boldsymbol{\rho})\overline{\mathbf{u}} = \mathbf{l}$.

Preferred solvers

How are the linear systems $\mathbf{K}(\rho)\mathbf{u} = \mathbf{f}$, $\mathbf{K}(\rho)\bar{\mathbf{u}} = \mathbf{l}$ solved?

- 2-D: Direct sparse solvers.
- 3-D: Iterative solvers, e.g. Krylov subspace solvers with preconditioning.
- Linear elasticity: Symmetric positive-definite stiffness matrix $\rightarrow$ Preconditioned Conjugate Gradients (PCG).
- Current study: Preconditioning achieved by applying multigrid solver $\rightarrow$ "MGCG".

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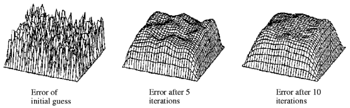
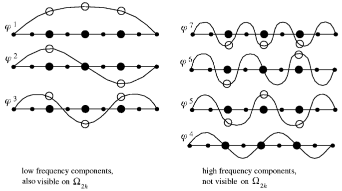
Brief history of multigrid applied in topopt: For solving KKT systems (Maar & Schultz 2000, Stainko 2006) ; multilevel approach (Stainko 2006) .

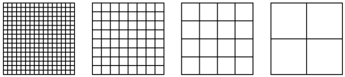
Multigrid principles 1

(from Trottenberg, Oosterlee and Schüller)

- Classical iterative methods smooth the error of an approximation.
- Smooth errors are well approximated on a coarser grid.
- Two-grid cycle = smoothing + coarse-grid correction.
- Recursive coarse-grid corrections → multigrid V-cycle.
- Multigrid methods developed since 1970's, considered optimal → number of operations $O(n)$.





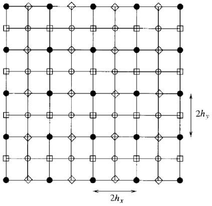


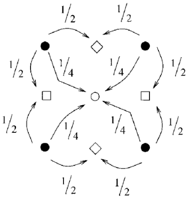
Multigrid principles 2

The two-grid algorithm $\mathbf{u} = \mathrm{MG}(\mathbf{u}, \mathbf{f}, \mathbf{K})$:

- Pre-smooth $\mathbf{u} = \mathbf{u} + \mathbf{S}^{-1}(\mathbf{f} - \mathbf{K}\mathbf{u})$
- Coarse grid correction - solve $\mathbf{K_c}\mathbf{u_c} = \mathbf{P}^{\mathsf{T}}(\mathbf{f} - \mathbf{K}\mathbf{u})$ with $\mathbf{K_c} = \mathbf{P}^{\mathsf{T}}\mathbf{K}\mathbf{P}$ (Galerkin projections)
- Interpolate $\mathbf{u} = \mathbf{u} + \mathbf{P}\mathbf{u_c}$
- Post-smooth $\mathbf{u} = \mathbf{u} + \mathbf{S}^{-1}(\mathbf{f} - \mathbf{K}\mathbf{u})$
- return $\mathbf{u}$

Prolongation operator:





MGCG principle

Idea of preconditioning - use a matrix or an operator $\mathbf{M}$:

$$\kappa\left(\mathbf{M}^{-1}\mathbf{K}\right) << \kappa\left(\mathbf{K}\right)$$

MGCG - a multigrid V-cycle replaces the action of $\mathbf{M}$.

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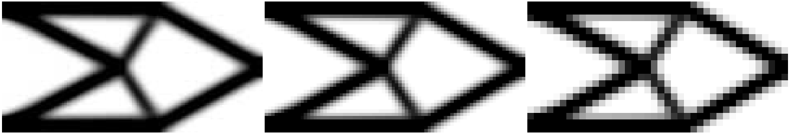
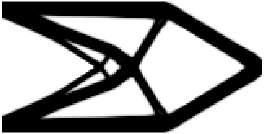
- Exploits the similarity of designs on different grid levels



- Challenge to capture small features on the fine grid

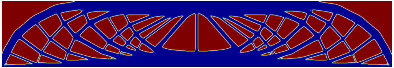


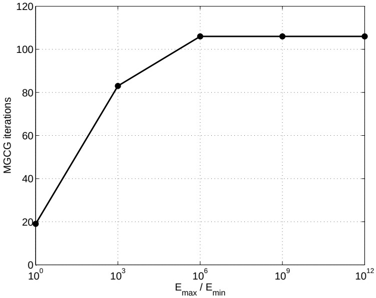
MGCG performance in topopt (1)

MGCG converges nicely even for high-contrast layouts:

- Test case 480×160 MBB beam (symm. half).
- 106 MGCG iterations for contrast 10^6 and above.
- With incomplete Cholesky preconditioner:

contrast	iterations
1	774
10^3	1,707
10^6	1,896



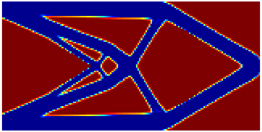


MGCG performance in topopt (2)

MGCG requires few iterations to reach accurate optimization outcome:

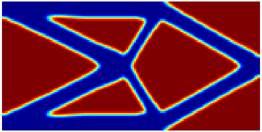
- Test case 160×80 cantilever, $r = 1.5$, contrast 10^9, sensitivity filter, OC.

MGCG it. per cycle	objective value	relative diff. (%)
3	77.45	-0.013
4	77.45	-0.013
5	77.47	+0.013
31.5	77.46	accurate



- Test case 160×80 cantilever, $r = 3.0$, contrast 10^9, density filter, MMA.

MGCG it. per cycle	objective value	relative diff. (%)
1	83.83	-0.12
2	83.85	-0.095
3	83.93	0.0
19.0	83.93	accurate



MGCG performance in topopt (3)

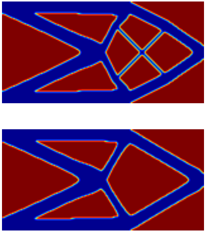
Monitoring the accuracy of the design sensitivities to avoid local minima:

$$\left|\widetilde{\overline{\mathbf{u}}}_k^{\mathsf{T}} \frac{\partial \mathbf{K}}{\partial \rho_e} \widetilde{\mathbf{u}}_k - \widetilde{\overline{\mathbf{u}}}_{k-1}^{\mathsf{T}} \frac{\partial \mathbf{K}}{\partial \rho_e} \widetilde{\mathbf{u}}_{k-1}\right| < \left| -\overline{\mathbf{u}}^{\mathsf{T}} \frac{\partial \mathbf{K}}{\partial \rho_e} \mathbf{u} + \Lambda v_e \right| \qquad \forall e \,|0 < \rho_e \; < 1$$

$$\frac{\left|\widetilde{\overline{\mathbf{u}}}_k^{\mathsf{T}} \frac{\partial \mathbf{K}}{\partial \rho_e} \widetilde{\mathbf{u}}_k - \widetilde{\overline{\mathbf{u}}}_{k-1}^{\mathsf{T}} \frac{\partial \mathbf{K}}{\partial \rho_e} \widetilde{\mathbf{u}}_{k-1}\right|}{\left|\widetilde{\overline{\mathbf{u}}}_k^{\mathsf{T}} \frac{\partial \mathbf{K}}{\partial \rho_e} \widetilde{\mathbf{u}}_k\right|} < \eta \qquad \forall e$$

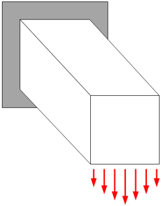
Test case 160×80 cantilever, $r = 1.5$, contrast 10^9, density filter, MMA:

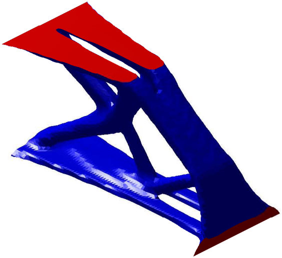
MGCG it. per cycle	objective value	relative diff. (%)
5 (fixed)	79.92	+1.6
6 (fixed)	78.67	+0.013
5.12 (monitored)	78.67	+0.013
7.42 (monitored)	78.65	-0.013
44.6	78.66	accurate

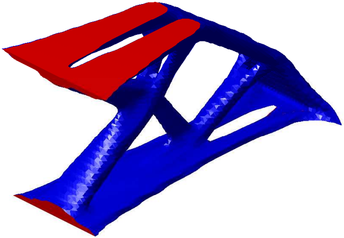


3-D minimum compliance

- $80 \times 40 \times 40$, $r = \sqrt{3}$, 4 MG levels, over 400,000 DOF, 50 design cycles in less than 15 minutes in MATLAB.
- $48 \times 24 \times 24$, $r = \sqrt{3}$, 4 MG levels, over 90,000 DOF, 50 design cycles in 3 minutes in MATLAB.

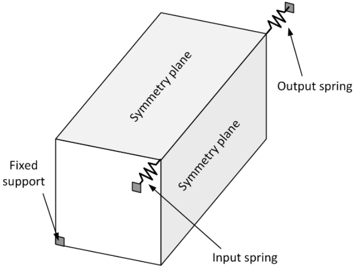


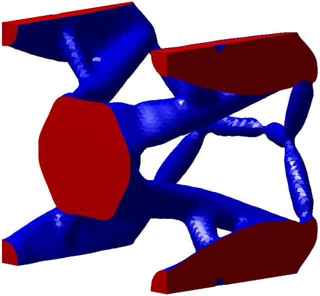




3-D force inverter

- $64 \times 32 \times 32$ mesh, $r = 3$, 4 MG levels, over 200,000 DOF, 100 design cycles in less than 24 minutes in MATLAB.
- MGCG applied as block-PCG - 2 r.h.s. simultaneously.





Conclusion

- MGCG performs very well - even for high-contrast stiffness distributions.
- Only few MGCG iterations are required for achieving accurate optimization.
- Accuracy is ensured by monitoring the design sensitivities.
- Paves the way for efficient implementations in standard computational environments: plug-ins for modeling software; applications on mobile devices.

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Funding:

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Further reading:

Amir O, Aage N and Lazarov BS. On multigrid-CG for efficient topology optimization, submitted (2-D code attached to the article.)